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A CHARACTERIZATION OF THE NORMAL DISTRIBUTION USING STATIONARY MAX-STABLE PROCESSES

SEBASTIAN ENGELKE AND ZAKHAR KABLUCHKO

Abstract. Consider the max-stable process $\eta(t) = \max_{i\in\mathbb{N}} U_i \mathrm{e}^{\langle \mathrm{X_i}, \mathrm{t}\rangle - \kappa(\mathrm{t})}$, $t \in \mathbb{R}^d$, where $\{U_i, i \in \mathbb{N}\}$ are points of the Poisson process with intensity $u^{-2}\mathrm{d}u$ on $(0,\infty)$, X_i, $i \in \mathbb{N}$, are independent copies of a random d-variate vector X (that are independent of the Poisson process), and $\kappa : \mathbb{R}^d \to \mathbb{R}$ is a function. We show that the process η is stationary if and only if X has multivariate normal distribution and $\kappa(t) - \kappa(0)$ is the cumulant generating function of X. In this case, η is a max-stable process introduced by R. L. Smith.

1. Introduction

The first class of max-stable processes that was used as a flexible model for spatially distributed extreme events was introduced by Smith (1990). Let N be a random vector having a zero-mean d-variate normal distribution with covariance matrix Σ. Further, let N_i, $i \in \mathbb{N}$, be independent copies of N. Independently of the N_i's, let $\{U_i, i \in \mathbb{N}\}$ be a Poisson point process on $(0,\infty)$ with intensity $u^{-2}\mathrm{d}u$. The stochastic process

$$M_\Sigma(t) := \max_{i\in\mathbb{N}} U_i \exp\left\{\langle N_i, t\rangle - \frac{1}{2}\langle t, \Sigma t\rangle\right\}, \quad t \in \mathbb{R}^d, \tag{1}$$

is now commonly termed the *Smith process*. This process is max-stable and stationary, where the former means that the pointwise maximum $\frac{1}{n}\max_{i=1}^n M_{\Sigma,i}$ of n independent copies $M_{\Sigma,1}, \dots, M_{\Sigma,n}$ of the process M_Σ has the same finite-dimensional distributions as M_Σ itself, for all $n \in \mathbb{N}$. Recall also that a stochastic process $\{M(t), t \in \mathbb{R}^d\}$ is said to be stationary if it has the same finite-dimensional distributions as the shifted process $\{M(t+h), t \in \mathbb{R}^d\}$, for all $h \in \mathbb{R}^d$.

Thanks to its simple form and the small number of parameters in low dimensions, the Smith process has become a widely applied model in spatial extreme value statistics (de Haan and Pereira, 2006; Engelke et al., 2015; Oesting, 2015; Padoan et al., 2010; Davison and Gholamrezaee, 2012; Westra and Sisson, 2011) and it has been extended in several directions (de Haan and Pereira, 2006; Kabluchko et al., 2009; Smith and Stephenson, 2009; Robert, 2013). Smith processes also appeared in connection with convex hulls of independent and identically distributed samples (Eddy and Gale, 1981; Hooghiemstra and Hüsler, 1996). It is worth noting that the class of Brown–Resnick processes introduced in Kabluchko et al. (2009) includes the Smith process as a special case. In fact, instead of the random parabolas $\langle N_i, t\rangle - \frac{1}{2}\langle t, \Sigma t\rangle$ in (1), it is possible to consider independent copies Z_i, $i \in \mathbb{N}$,

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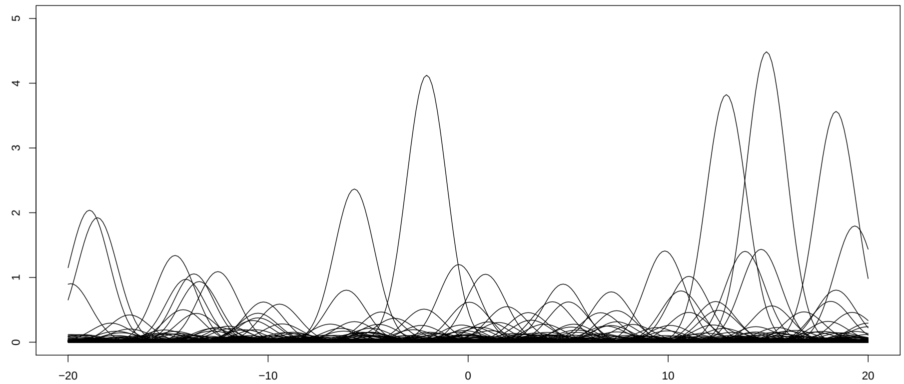

FIGURE 1. Realization of the one-dimensional Smith process (1).

of a zero-mean Gaussian process $\{Z(t), t \in \mathbb{R}^d\}$ with stationary increments and variance function $\sigma^2(t) = \operatorname{Var} Z(t)$. The Brown–Resnick process (cf., Brown and Resnick, 1977; Kabluchko et al., 2009)

$$M_{\mathrm{BR}}(t) := \max_{i \in \mathbb{N}} U_i \exp\left\{ Z_i(t) - \frac{1}{2}\sigma^2(t) \right\}, \quad t \in \mathbb{R}^d, \tag{2}$$

is stationary, max-stable and its distribution depends only on the so-called variogram $\gamma(t) = \operatorname{Var}(Z(t) - Z(0))$. Choosing Z to be the random linear Gaussian function $Z(t) = \langle N, t\rangle$ which has $\sigma^2(t) = \langle t, \Sigma t\rangle$, one recovers the Smith process.

In the case when the matrix Σ is non-singular, the Smith process M_Σ defined in (1) can equivalently be represented as a moving maxima process (cf., Wang and Stoev, 2010, for instance)

$$M_\Sigma(t) = \frac{\det(\Sigma)^{1/2}}{(2\pi)^{d/2}} \max_{i \in \mathbb{N}} V_i \exp\left\{ -\frac{1}{2}\langle (t - T_i), \Sigma(t - T_i)\rangle \right\}. \tag{3}$$

Here, $\{(T_i, V_i), i \in \mathbb{N}\}$ is a Poisson point process with intensity $\mathrm{d}t \times v^{-2}\mathrm{d}v$ on $\mathbb{R}^d \times (0, \infty)$. In the moving maxima representation (3), M_Σ can be interpreted as the maximum over many "storms" where the i-th storm has center point T_i and strength V_i, and all storms have a common spatial shape given by the d-variate normal density with covariance matrix Σ^{-1} (as opposed to Σ in representation (1)). Figure 1 shows a one-dimensional stationary Smith process. Note in particular that because of stationarity the origin does not play an exceptional role.

In this note we investigate the following question. If we drop the assumption of Gaussianity in (1) and replace the normally distributed random vector N by an arbitrary d-dimensional vector X, can we find a function $\kappa : \mathbb{R}^d \to \mathbb{R}$ such that the stochastic process

$$\eta(t) := \max_{i \in \mathbb{N}} U_i \exp\{\langle X_i, t\rangle - \kappa(t)\}, \quad t \in \mathbb{R}^d, \tag{4}$$

is max-stable and stationary? Here, $\{U_i, i \in \mathbb{N}\}$ is a Poisson point process on $(0, \infty)$ with intensity $u^{-2}\mathrm{d}u$, and X_i, $i \in \mathbb{N}$, are independent copies of X which are also independent of the Poisson process. In fact, the max-stability of the process η

follows directly by the construction since (4) is the de Haan representation (de Haan, 1984) of the process η. The crucial part of the question above is the stationarity.

2. Main result

Theorem 2.1. *The process η defined in* (4) *is stationary on $\mathbb{R}^d$ if and only if X has a d-variate normal distribution with some mean $\mu \in \mathbb{R}^d$ and covariance matrix Σ, and*

$$\kappa(t) = \langle \mu, t\rangle + \frac{1}{2}\langle t, \Sigma t\rangle + \kappa(0). \tag{5}$$

Proof. If X is d-variate normal with mean μ and covariance matrix Σ, and $\kappa(t)$ is given by (5), then it is well known that η is stationary; see for example Theorem 2 in Kabluchko et al. (2009).

Let η be stationary. We have to show that X is normal and (5) holds. Without restriction of generality let $\kappa(0) = 0$ because otherwise we could replace $\kappa(t)$ by $\kappa(t) - \kappa(0)$ without changing the stationarity of η. Then, $\eta(0)$ has unit Fréchet distribution, i.e.,

$$\mathbb{P}(\eta(0) \le x) = \mathbb{P}(\max_{i\in\mathbb{N}} U_i \le x) = \exp\{-1/x\},$$

for $x > 0$. The stationarity of the one-dimensional margins of η implies that the cumulant generating function

$$\varphi(t) := \log \mathbb{E}\mathrm{e}^{\langle X,t\rangle}, \quad t \in \mathbb{R}^d, \tag{6}$$

is finite, and that $\kappa(t) = \varphi(t)$, for all $t \in \mathbb{R}^d$. Indeed,

$$\begin{aligned}\mathbb{P}(\eta(t) \le x) &= \mathbb{P}\left(\max_{i\in\mathbb{N}} U_i \mathrm{e}^{\langle X_i,t\rangle - \kappa(t)} \le x\right)\\ &= \exp\left\{-\int_0^\infty \mathbb{P}\left(\mathrm{e}^{\langle X,t\rangle-\kappa(t)} > \frac{x}{u}\right)\frac{\mathrm{d}u}{u^2}\right\}\\ &= \exp\left\{-\frac{1}{x}\,\mathbb{E}\mathrm{e}^{\langle X,t\rangle-\kappa(t)}\right\}.\end{aligned}$$

Thus, in fact, $\kappa(t) = \varphi(t)$ for all $t \in \mathbb{R}^d$.

For $n \in \mathbb{N}$ and $t_1, \dots, t_n \in \mathbb{R}^d$, the cumulant generating function $\varphi_{t_1,\dots,t_n}$ of the n-variate random vector $(\langle X, t_1\rangle - \varphi(t_1), \dots, \langle X, t_n\rangle - \varphi(t_n))$ is given by

$$\varphi_{t_1,\dots,t_n}(u_1,\dots,u_n) := \log \mathbb{E}\exp\left\{\sum_{i=1}^n (\langle X, t_i\rangle - \varphi(t_i))u_i\right\} \tag{7}$$

$$= \varphi\left(\sum_{i=1}^n u_i t_i\right) - \sum_{i=1}^n u_i \varphi(t_i), \tag{8}$$

for all $u_1, \dots, u_n \in \mathbb{R}$. Therefore, the general stationarity criterion for max-stable processes of the form (4) given in Proposition 6 in Kabluchko et al. (2009) implies that η is stationary, if and only if for all $h \in \mathbb{R}^d$ and all $u_1, \dots, u_n \in [0, 1]$ such that $\sum_{i=1}^n u_i = 1$ it holds that

$$\varphi_{t_1,\dots,t_n}(u_1,\dots,u_n) = \varphi_{t_1+h,\dots,t_n+h}(u_1,\dots,u_n), \tag{9}$$

or, by (7),

$$\varphi\left(\sum_{i=1}^n u_i t_i\right) - \sum_{i=1}^n u_i \varphi(t_i) = \varphi\left(h + \sum_{i=1}^n u_i t_i\right) - \sum_{i=1}^n u_i \varphi(t_i + h). \tag{10}$$

For arbitrary $t_1, t_2, h \in \mathbb{R}^d$ and $\delta \in [0,1]$ we have by (10) that

$$\begin{aligned}\varphi\left((1-\delta)t_1 + \delta t_2\right) &- (1-\delta)\varphi(t_1) - \delta\varphi(t_2) \\ &= \varphi\left((1-\delta)t_1 + \delta t_2 + h\right) - (1-\delta)\varphi(t_1+h) - \delta\varphi(t_2+h).\end{aligned}$$

Applying the above relation with εh instead of h and rearranging the terms, we obtain that for every $\varepsilon > 0$,

$$\begin{aligned}&\frac{\varphi\left((1-\delta)t_1 + \delta t_2 + \varepsilon h\right) - \varphi\left((1-\delta)t_1 + \delta t_2\right)}{\varepsilon} \\ &\qquad = (1-\delta)\frac{\varphi(t_1+\varepsilon h) - \varphi(t_1)}{\varepsilon} + \delta\frac{\varphi(t_2+\varepsilon h) - \varphi(t_2)}{\varepsilon}.\end{aligned}$$

Note that the function φ is infinitely differentiable by its definition (6). Sending $\epsilon \searrow 0$ on both sides gives

$$\langle \nabla\varphi\left((1-\delta)t_1 + \delta t_2\right), h\rangle = (1-\delta)\langle\nabla\varphi(t_1), h\rangle + \delta\langle\nabla\varphi(t_2), h\rangle, \tag{11}$$

where $\nabla\varphi(t) \in \mathbb{R}^d$ denotes the gradient of φ at $t \in \mathbb{R}^d$. Since equation (11) holds for all $h \in \mathbb{R}^d$, we obtain

$$\nabla\varphi\left((1-\delta)t_1 + \delta t_2\right) = (1-\delta)\nabla\varphi(t_1) + \delta\nabla\varphi(t_2), \tag{12}$$

for all $t_1, t_2 \in \mathbb{R}^d$ and $\delta \in [0,1]$. Thus, any of the d components of the gradient $\nabla\varphi$ is both convex and concave and hence, an affine function. Consequently, there is a $d \times d$ matrix $\Sigma \in \mathrm{Mat}_d(\mathbb{R})$ and a vector $\mu \in \mathbb{R}^d$ such that

$$\nabla\varphi(t) = \Sigma t + \mu.$$

Observe that the function $\xi : \mathbb{R}^d \to \mathbb{R}$, $\xi(t) = \langle\mu, t\rangle + \frac{1}{2}\langle t, \Sigma t\rangle$, $t \in \mathbb{R}^d$, has the same gradient as φ. So, the difference between these two functions has gradient 0 and we conclude that this difference is constant and, in fact, 0 by the assumption $\varphi(0) = \kappa(0) = 0$. We obtain that

$$\kappa(t) = \varphi(t) = \log \mathbb{E}\exp\{\langle X, t\rangle\} = \langle\mu, t\rangle + \frac{1}{2}\langle t, \Sigma t\rangle \quad t \in \mathbb{R}^d.$$

It remains to observe that the matrix Σ must be positive semidefinite because the function φ, being a cumulant generating function, is convex. Thus, X has a d-variate normal distribution with mean vector μ and covariance matrix Σ, which completes the proof. □

Remark 2.2. For any random vector X it is possible to make the one-dimensional distributions of the process η defined in (4) stationary by choosing $\kappa(t)$ to be the cumulant generating function of X. However, the above proof shows that except in the case when X is normal, it is not possible to make the two-dimensional distributions of η stationary.

Remark 2.3. Similarly to the setting of the present paper one may ask whether in the definition of the Brown–Resnick process (2) we can replace Z by some more general process (for example, a Gaussian process with non-stationary increments). This question was studied in Kabluchko (2010).

Remark 2.4. Theorem 2.1 provides a characterization of the multivariate normal distribution based on max-stable processes. See (Kotz et al., 2000, page 151) for a review of known characterizations of the multivariate normal law.

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Sebastian Engelke, Ecole Polytechnique Fédérale de Lausanne, EPFL-FSB-MATHAA-STAT, Station 8, 1015 Lausanne, Switzerland
E-mail address: sebastian.engelke@epfl.ch

Zakhar Kabluchko, Institute of Mathematical Statistics, University of Münster, Orléans-Ring 10, 48149 Münster, Germany
E-mail address: zakhar.kabluchko@uni-muenster.de